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On the Quasi-stationary Maxwell Equations with Monotone Characteristics in a Multiply Connected Domain*

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The quasi-stationary Maxwell equations for a magnetic core with a non-linear characteristic and homeomorphic to a three dimensional torus are reduced to a unique evolution equation involving a monotone operator. This reduction is based on the properties of some functional spaces suited to the study of the operators "curl" and "div," with different kinds of boundary and period conditions. Some existence, uniqueness and regularity results are proved.

1. INTRODUCTION

In the study of electromagnetic devices, displacement currents are usually neglected, so that the Maxwell equations reduce to a parabolic type system. We are here particularly concerned with the magnetic field distribution in transformer cores, and we study the simplest case of a core homeomorphic to a three-dimensional torus, with a given electromotive force $\dot{\omega}(t)$ controlling the flux. More complex kinds of domains can be treated in the same way. Hysteresis phenomena are neglected, so that the non-linear magnetic characteristic ζ can be assumed of monotone type with asymptotically linear behavior. We also assume that eddy currents are everywhere present, that is, the conductivity σ is strictly positive.

In Section 1 the problem is formulated in terms of the usual vector potential for the magnetic induction and a scalar potential for the electric field; in Section 2 several functional spaces are introduced to handle the differential operators "div" and "curl" and to take into account various kinds of boundary and periods conditions. The orthogonal decomposition of $(L^2(\Omega))^3$ into divergence free vector functions and gradients (Foias and Temam, [6]) is used in Section 3 to reduce the problem to a unique evolution equation with a monotone operator, for which existence, uniqueness and

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regularity results are derived. In Section 4 a direct approach to the problem, which could be more useful for the numerical treatment, is sketched. In Negro [10], a similar problem was studied, introducing a vector potential for the current density and a scalar potential for the magnetic field, with time periodic conditions, and allowing the electric conductivity to vanish.

2. The Differential Equations and the Electromagnetic Potentials

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with a smooth boundary $\partial\Omega$, homeomorphic to a bidimensional torus; let Γ and γ be, respectively, a relative 2-cycle and an absolute 1-cycle in Ω not homologous to 0; let $\mathbf{n}$ be the outward normal to $\partial\Omega$, and ν the normal to Γ, oriented as in Fig. 1.

We consider the following quasi-stationary Maxwell equations:

$$\begin{aligned} &j - \operatorname{curl} H = 0 \\ &H = \zeta(B) \qquad \text{in } \Omega; \\ &\operatorname{div} B = 0 \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\mathbf{n} \cdot B = 0 \qquad \text{on } \partial\Omega; \\ &\int_\Gamma \nu \cdot B = \omega \end{aligned} \tag{1.2}$$

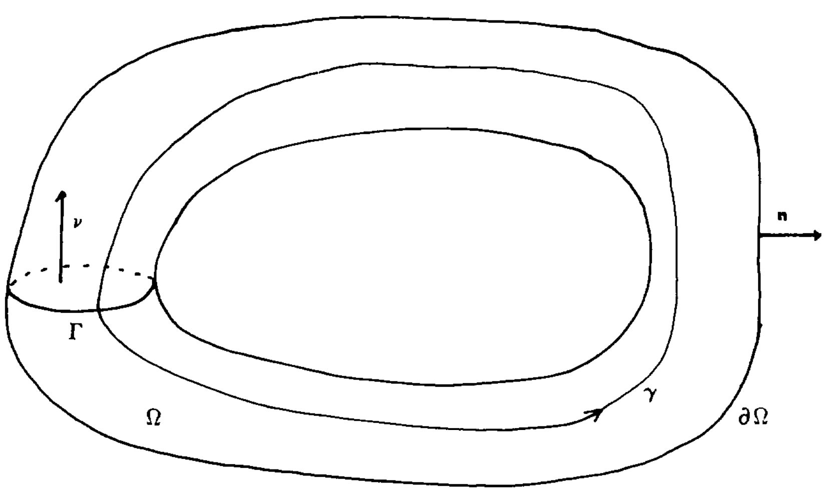

Fig. 1

$$
\begin{aligned}
&\partial B/\partial t + \operatorname{curl} E = 0 \\
&j = \sigma E \qquad \text{in } \Omega; \\
&\operatorname{div} j = 0
\end{aligned}
\tag{1.3}
$$

$$
\begin{aligned}
&\mathbf{n} \cdot j = 0 \qquad \text{on } \partial\Omega; \\
&\int_\Gamma \nu \cdot j = 0 \\
&H(0) = H_0.
\end{aligned}
\tag{1.4}
$$

We transform these equations recurring to the usual vector and scalar potentials: the flux $\omega(t)$ being given, there exists, by Hodge's theorem (Hodge, [8, III, 30.1]), a unique harmonic field $\alpha(t)$ on $\partial\Omega$ satisfying

$$d\alpha = 0, \ \delta\alpha = 0; \qquad \int_\gamma \alpha = 0; \qquad \int_{\partial\Gamma} \alpha = \omega \tag{1.5}$$

(here d and δ are the differential and codifferential operators on $\partial\Omega$); we can then determine (Duff, [3, III]) a unique field A in Ω such that

$$
\begin{aligned}
&\operatorname{curl} A = B, \\
&\operatorname{div} A = 0, \\
&\mathbf{n} \times A = \mathbf{n} \times \alpha = \tilde{\alpha}
\end{aligned}
\tag{1.6}
$$

(in fact, regarding B as a 2-form and A as a 1-form, it is easily seen that the required compatibility conditions are satisfied. Moreover, the only harmonic 1-field in Ω with null tangential component is 0). Since then

$$\operatorname{curl}(\partial A/\partial t + E) = 0 \tag{1.7}$$

and $\Omega' = \Omega \backslash \Gamma$ is simply connected, there exists a scalar potential φ such that

$$\partial A/\partial t + E = \nabla\varphi \qquad \text{in } \Omega'; \tag{1.8}$$

and since by (1.7) the tangential component of $\nabla\varphi$ is continuous across Γ, it comes that

$$\mathscr{S}(\varphi) = \varphi|_{\Gamma^+} - \varphi|_{\Gamma^-}$$

is a real constant (Γ^+ and Γ^- are the upper and lower sheets of Γ with respect to ν). We have then

$$j = \sigma(\nabla\varphi - \partial A/\partial t)$$

and therefore

$$\operatorname{div} \sigma(\nabla\varphi - \partial A/\partial t) = 0,$$

$$\mathbf{n} \cdot \sigma(\nabla\varphi - \partial A/\partial t) = 0, \tag{1.9}$$

$$\int_\Gamma \nu \cdot \sigma(\nabla\varphi - \partial A/\partial t) = 0;$$

moreover, Eq. (1.1) can be restated as

$$\sigma(\nabla\varphi - \partial A/\partial t) - \operatorname{curl} H = \sigma(\nabla\varphi - \partial A/\partial t) - \operatorname{curl} \zeta(\operatorname{curl} A) = 0,$$

that is,

$$\sigma\, \partial A/\partial t + \operatorname{curl} \zeta(\operatorname{curl} A) = \sigma\nabla\varphi. \tag{1.10}$$

3. Some Functional Spaces

This section deals with several functional spaces which occur in the study of electromagnetism problems. As a standard notation, if E is a space, we shall write $\mathbb{E}$ for E^3, and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between E' and E. In the following, Ω will be an open set such as described in Section 1.

Our main references are the Hilbert spaces $\mathbb{L}^2(\Omega)$ and $H^1(\Omega)$, of which we shall define some subspaces and their relating properties. We set (Duvaut and Lions [5, VII, 4]):

$$H(\operatorname{curl}, \Omega) = \{u \in \mathbb{L}^2(\Omega) | \operatorname{curl} u \in \mathbb{L}^2(\Omega)\},$$

$$H(\operatorname{div}, \Omega) = \{u \in \mathbb{L}^2(\Omega) \mid \operatorname{div} u \in L^2(\Omega)\}.$$

They are Hilbert spaces with respect to the graph norm, dense in $\mathbb{L}^2(\Omega)$; for vector functions in such spaces, the tangential and normal components $\mathbf{n} \times u$ and $\mathbf{n} \cdot u$ can, respectively, be defined in $\mathbb{H}^{-1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$, allowing us to consider the closed subspaces (dense in $\mathbb{L}^2(\Omega)$):

$$H_0(\operatorname{curl}, \Omega) = \{u \in H(\operatorname{curl}, \Omega) \mid \mathbf{n} \times u = 0\},$$

$$H_0(\operatorname{div}, \Omega) = \{u \in H(\operatorname{div}, \Omega) \mid \mathbf{n} \cdot u = 0\}.$$

The following relations then hold.

(a) (Integration by parts theorem):

$$\langle \mathbf{n} \times f, \varphi \rangle_{\partial\Omega} = (\operatorname{curl} f, \Phi) - (f, \operatorname{curl} \Phi),^1 \tag{2.1}$$

$$\langle \mathbf{n} \cdot g, \psi \rangle_{\partial\Omega} = (\operatorname{div} g, \Psi) + (g, \nabla\Psi), \tag{2.2}$$

[1] When duality pairings are considered for functions on different boundaries, the actual boundary will be explicitly denoted.

valid for all $f \in H(\mathrm{curl}, \Omega)$, $g \in H(\mathrm{div}, \Omega)$; $\Phi \in \mathbb{H}^1(\Omega)$, $\Phi|_{\partial\Omega} = \varphi \in \mathbb{H}^{1/2}(\partial\Omega)$; $\Psi \in H^1(\Omega)$, $\Psi|_{\partial\Omega} = \psi \in H^{1/2}(\partial\Omega)$.

(b) (Friedrichs [7, 10]; Duvaut and Lions [5, VII]):

$$H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) = \{u \in \mathbb{H}^1(\Omega) \mid \mathbf{n} \times u = 0\},$$
$$H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega) = \{u \in \mathbb{H}^1(\Omega) \mid \mathbf{n} \cdot u = 0\}.$$

We shall consider the space

$$V_0 = \{u \in H_0(\mathrm{curl}, \Omega) \mid \mathrm{div}\, u = 0\},$$

which is therefore a closed subspace of $\mathbb{H}^1(\Omega)$, and can be equipped with the equivalent Hilbert norm

$$\|u\|_{V_0} = \|\mathrm{curl}\, u\|_{\mathbb{L}^2(\Omega)}$$

because of Friedrichs' inequality (Friedrichs [7]) and the non-existence of relative 1-cycles in Ω not homologous to 0.

Given a harmonic field α on $\partial\Omega$, we shall also consider the closed affine submanifold of $H(\mathrm{curl}, \Omega)$

$$V_\alpha = \{u \in H(\mathrm{curl}, \Omega) \mid \mathrm{div}\, u = 0, \mathbf{n} \times u = \tilde{\alpha}\},$$

which is a translation of V_0. We consider then the closed subspaces of $\mathbb{L}^2(\Omega)$,

$$H = \{u \in \mathbb{L}^2(\Omega) \mid \mathrm{div}\, u = 0\},$$
$$K = \{u \in \mathbb{L}^2(\Omega) \mid \mathrm{curl}\, u = 0\},$$

and refer to the orthogonal decomposition of $\mathbb{L}^2(\Omega)$ (Foias and Temam [6, 1.2, 1.1]):

$$\mathbb{L}^2(\Omega) = H_0 \oplus K,$$

where $H_0 = \{u \in H \mid \mathbf{n} \cdot u = 0, \int_\Gamma \nu \cdot u = 0\}$.

We call M and $N = I - M$ the projections of $\mathbb{L}^2(\Omega)$ onto K and H_0, respectively: both M and N are obviously symmetric and positive contractions. We then define

$$W = \{u \in H_0 \mid \mathrm{curl}\, u \in H_0\},$$

a Hilbert space with respect to the norm

$$\|u\|_W = \|\mathrm{curl}\, u\|_{\mathbb{L}^2(\Omega)},$$

and eventually

$$\mathscr{U} = \left\{ u \in H^1(\Omega') \mid \mathscr{L}(u) \in \mathbb{R}, \int_\Omega u = 0 \right\}, \qquad (\Omega' = \Omega \backslash \Gamma),$$

which, because of the 0-mean value constraint, is a Hilbert space with respect to the norm

$$\|u\|_{\mathscr{U}} = \|\nabla u\|_{\mathbb{L}^2(\Omega)};$$

$$L^2_m(\Omega) = \left\{ u \in L^2(\Omega) \;\middle|\; \int_\Omega u = 0 \right\}.$$

We now claim that:

PROPOSITION I. *V_0 is dense in H.*

PROPOSITION II. *$\mathscr{U}$ is dense in $L^2_m(\Omega)$.*

PROPOSITION III. $K = \{u \in \mathbb{L}^2(\Omega) \mid u = \nabla p, \, p \in \mathscr{U}\}$.

PROPOSITION IV. *N is an isomorphism between V_0 and W.*

PROPOSITION V. *W is dense in H_0.*

Proof of Proposition I. $\forall \mathbf{v} \in H$, let $\mathbf{v_n}$ be a sequence in $H_0(\text{curl}, \Omega)$ such that $\mathbf{v_n} \to \mathbf{v}$ in $\mathbb{L}^2(\Omega)$. Define, for all n, $\mathbf{p_n}$ as the unique solution of

$$\mathbf{p_n} \in H^1_0(\Omega),$$

$$-\Delta \mathbf{p_n} = \text{div}\, \mathbf{v_n}$$

and $\mathbf{w_n} = \mathbf{v_n} + \nabla \mathbf{p_n}$. Then $\mathbf{w_n} \in V_0$, since: $\text{curl}\, \mathbf{w_n} = \text{curl}\, \mathbf{v_n} \in \mathbb{L}^2(\Omega)$; $\text{div}\, \mathbf{w_n} = \text{div}\, \mathbf{v_n} + \Delta \mathbf{p_n} = 0$; and $\forall \psi \in C^\infty(\bar{\Omega})$: $\langle \mathbf{n} \times \nabla \mathbf{p_n}, \psi \rangle_{\partial\Omega} = (\text{curl}\, \nabla \mathbf{p_n}, \psi) - (\nabla \mathbf{p_n}, \text{curl}\, \psi) = (\mathbf{p_n}, \text{div curl}\, \psi) - \langle \mathbf{n} \cdot \text{curl}\, \psi, \mathbf{p_n} \rangle_{\partial\Omega} = 0$, and hence $\mathbf{n} \times \mathbf{w_n} = \mathbf{n} \times \nabla \mathbf{p_n} = 0$. Moreover, since $-\Delta \mathbf{p_n} = \text{div}\, \mathbf{v_n} \to \text{div}\, \mathbf{v} = 0$ in $H^{-1}(\Omega)$, then $\mathbf{p_n} \to 0$ in $H^1_0(\Omega)$ and so $\nabla \mathbf{p_n} \to 0$ in $\mathbb{L}^2(\Omega)$, so that $\mathbf{w_n} \to \mathbf{v}$ in H.

Proof of Proposition II. In fact, $\mathscr{U} \supseteq \{u \in \mathscr{U} \mid \mathscr{L}(u) = 0\} = H^1_m(\Omega) = \{u \in H^1(\Omega) \mid \int_\Omega u = 0\}$, which is dense in $L^2_m(\Omega)$: if $\mathbf{u} \in L^2_m(\Omega)$, let $\mathbf{u_n}$ be a sequence in $H^1(\Omega)$ such that $\mathbf{u_n} \to \mathbf{u}$ in $L^2(\Omega)$. Then, setting $\mathbf{c_n} = \mu(\Omega)^{-1} \int_\Omega \mathbf{u_n}$, $\mathbf{v_n} = \mathbf{u_n} - \mathbf{c_n} \in H^1_m(\Omega)$; moreover, $\int_\Omega \mathbf{u_n} \to \int_\Omega \mathbf{u} = 0$, so that $\mathbf{c_n} \to 0$ and $\mathbf{v_n} \to \mathbf{u}$ in $L^2_m(\Omega)$.

Proof of Proposition III. If $\mathbf{u} = \nabla \mathbf{p}$ for some $\mathbf{p} \in \mathscr{U}$, let $\psi \in (\mathscr{D}(\Omega))^3$. Then $\langle \text{curl}\, \mathbf{u}, \psi \rangle = \int_\Omega \mathbf{u} \cdot \text{curl}\, \psi = \int_{\Omega'} \mathbf{u} \cdot \text{curl}\, \psi = \int_{\Omega'} \nabla \mathbf{p} \cdot \text{curl}\, \psi =$

$-\int_{\Omega'} \mathbf{p} \cdot \operatorname{div} \operatorname{curl} \psi + \int_{\partial\Omega'} (\mathbf{p}, \mathbf{n} \cdot \operatorname{curl} \psi) = \langle \mathbf{p}|_{\Gamma^+} - \mathbf{p}|_{\Gamma^-}, \ \nu \cdot \operatorname{curl} \psi \rangle_\Gamma = \mathscr{L}(\mathbf{p}) \int_\Gamma \nu \cdot \operatorname{curl} \psi = 0$ by Stokes' theorem. Conversely, if $\mathbf{u} \in K$, let $\mathbf{f} \in \mathscr{U}'$ be defined by $\langle \mathbf{f}, \psi \rangle = (\mathbf{u}, \nabla\psi)$, $\forall \psi \in \mathscr{U}$.

The coercive problem

$$\mathbf{p} \in \mathscr{U},$$
$$(\nabla \mathbf{p}, \nabla \psi) = \langle \mathbf{f}, \psi \rangle \qquad \forall \psi \in \mathscr{U}$$

has a unique solution $\mathbf{p}$: we claim that $\mathbf{z} = \mathbf{u} - \nabla \mathbf{p} = 0$. In fact, taking at first $\psi \in C_0^\infty(\Omega)$, and remarking that $\psi = \hat{\psi} + \mathbf{c}$, where $\hat{\psi} \in \mathscr{U}$, $\mathbf{c} \in \mathbb{R}$ and $\nabla\hat{\psi} = \nabla\psi$, we have: $\langle \operatorname{div} \mathbf{z}, \psi \rangle = -(\mathbf{z}, \nabla\psi) = -(\mathbf{z}, \nabla\hat{\psi}) = 0$, so that $\operatorname{div} \mathbf{z} = 0$ in $L^2(\Omega)$. Taking then $\psi \in C^\infty(\bar{\Omega})$, we have $\langle \mathbf{n} \cdot \mathbf{z}, \psi \rangle_{\partial\Omega} = (\operatorname{div} \mathbf{z}, \psi) + (\mathbf{z}, \nabla\psi) = 0$, so that $\mathbf{n} \cdot \mathbf{z} = 0$. Taking at least $\psi \in C^\infty(\bar{\Omega}')$ such that $\mathscr{L}(\psi) \in \mathbb{R}$, we have $\langle \nu \cdot \mathbf{z}, \mathscr{L}(\psi) \rangle_\Gamma = (\operatorname{div} \mathbf{z}, \psi) + (\mathbf{z}, \nabla\psi) - \langle \mathbf{n} \cdot \mathbf{z}, \psi \rangle_{\partial\Omega} = 0$, so that $\int_\Gamma \nu \cdot \mathbf{z} = 0$: therefore, $\mathbf{z} \in H_0$. But $\operatorname{curl} \nabla \mathbf{p} = 0$ in Ω and $\mathbf{u} \in K$, and thus $\operatorname{curl} \mathbf{z} = 0$ in Ω, so that $\mathbf{z} \in K$: hence, $\mathbf{z} = 0$.

Proof of Proposition IV. If $\mathbf{v} \in V_0$, of course $N\mathbf{v} \in H_0$; besides, $\operatorname{curl} N\mathbf{v} = \operatorname{curl} \mathbf{v} - \operatorname{curl} M\mathbf{v} = \operatorname{curl} \mathbf{v} \in \mathbb{L}^2(\Omega)$; $\operatorname{div} \operatorname{curl} N\mathbf{v} = 0$; to prove that $\mathbf{n} \cdot \operatorname{curl} N\mathbf{v} = 0$, let $\psi \in H^{1/2}(\partial\Omega)$ and $\Psi \in H^1(\Omega)$ such that $\Psi|_{\partial\Omega} = \psi$: then $\langle \mathbf{n} \cdot \operatorname{curl} \mathbf{v}, \psi \rangle_{\partial\Omega} = (\operatorname{div} \operatorname{curl} \mathbf{v}, \Psi) + (\operatorname{curl} \mathbf{v}, \nabla\Psi) = (\mathbf{v}, \operatorname{curl} \nabla\Psi) + \langle \mathbf{n} \times \mathbf{v}, \nabla\psi \rangle_{\partial\Omega} = 0$. To prove at last that $\int_\Gamma \nu \cdot \operatorname{curl} \mathbf{v} = 0$, consider in a region $\Omega^* \subseteq \Omega$ bounded by Γ, $\partial\Omega$ and a second 2-cycle Γ' relatively homologous to Γ a regular function θ such that $\theta|_\Gamma = 1$ and $\theta|_{\Gamma'} = 0$. We have: $\int_\Gamma \nu \cdot \operatorname{curl} \mathbf{v} = \int_{\partial\Omega^*} (\mathbf{n} \cdot \operatorname{curl} \mathbf{v}, \theta) = \int_{\Omega^*} \operatorname{div} \operatorname{curl} \mathbf{v} \cdot \theta + \int_{\Omega^*} \operatorname{curl} \mathbf{v} \cdot \nabla\theta = \int_{\Omega^*} \mathbf{v} \cdot \operatorname{curl} \nabla\theta + \int_{\partial\Omega^*} (\mathbf{n} \times \mathbf{v}, \nabla\theta) = \int_{\Gamma \cup \Gamma'} (\mathbf{n} \times \mathbf{v}, \nabla\theta) = 0$. Therefore, $NV_0 \subseteq W$. (See Fig. 2) Conversely, if $\mathbf{w} \in W$, let $\mathbf{f} \in V_0'$ be defined by $\langle \mathbf{f}, \mathbf{v} \rangle = (\operatorname{curl} \mathbf{w}, \operatorname{curl} \mathbf{v})$ $\forall \mathbf{v} \in V_0$. (See Fig. 2.)

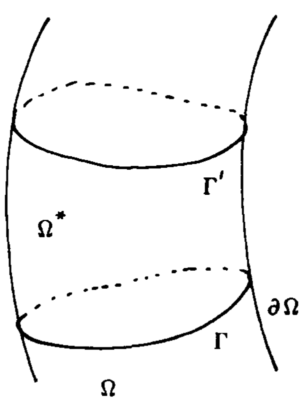

FIG. 2

The coercive problem

$$\mathbf{u} \in V_0,$$
$$(\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in V_0$$

has then a unique solution $\mathbf{u}$, and $N\mathbf{u} \in W$. To prove that $N\mathbf{u} = \mathbf{w}$, it is sufficient to show that $\operatorname{curl}\operatorname{curl}(N\mathbf{u} - \mathbf{w}) = \operatorname{curl}\operatorname{curl}(\mathbf{u} - \mathbf{w}) = 0$: in fact, $N\mathbf{u} - \mathbf{w} \in W$, so that both $N\mathbf{u} - \mathbf{w}$ and $\operatorname{curl}(N\mathbf{u} - \mathbf{w}) \in H_0$, in which all the harmonic fields are null (Duff [4, VI]), and hence if $\operatorname{curl}\operatorname{curl}(N\mathbf{u} - \mathbf{w}) = 0$, so are $\operatorname{curl}(N\mathbf{u} - \mathbf{w})$ and $N\mathbf{u} - \mathbf{w}$. Let therefore $\varphi \in (\mathscr{D}(\Omega))^3$ and $\mathbf{p} \in H_0^1(\Omega)$ such that $-\Delta \mathbf{p} = \operatorname{div} \varphi$: since $\mathbf{n} \times \nabla \mathbf{p} = 0$, $\varphi + \nabla \mathbf{p} \in V_0$ and $\langle \operatorname{curl}\operatorname{curl}(\mathbf{u} - \mathbf{w}), \varphi \rangle = (\operatorname{curl}(\mathbf{u} - \mathbf{w}), \operatorname{curl} \varphi) = (\operatorname{curl}(\mathbf{u} - \mathbf{w}), \operatorname{curl}(\varphi + \nabla \mathbf{p})) = 0$, so that $\operatorname{curl}\operatorname{curl}(\mathbf{u} - \mathbf{w}) = 0$ in $(\mathscr{D}'(\Omega))^3$. Remarking that eventually

$$\|\mathbf{u}\|_{V_0} = \|\operatorname{curl} \mathbf{u}\|_{\mathbb{L}^2(\Omega)} = \|N\mathbf{u}\|_W,$$

we conclude that N is an isomorphism.

Remark. We can therefore consider the restriction $N_0 = N|_{V_0}$, and the inverse operator N_0^{-1}: $W \to V_0$; we recall that

$$\|\mathbf{u}\|_W = \|N N_0^{-1} \mathbf{u}\|_W = \|N_0^{-1} \mathbf{u}\|_{V_0} = \|\operatorname{curl} \mathbf{u}\|_{\mathbb{L}^2(\Omega)};$$

and moreover, N being a projection in $\mathbb{L}^2(\Omega)$, $N = N^2$.

Proof of Proposition V. $\forall \mathbf{v} \in H_0 \subseteq H$, let $\mathbf{v}_n$ be a sequence in V_0 such that $\mathbf{v}_n \to \mathbf{v}$ in $\mathbb{L}^2(\Omega)$ (by Proposition I): then $\mathbf{w}_n = N\mathbf{v}_n \to N\mathbf{v} = \mathbf{v}$ in $\mathbb{L}^2(\Omega)$, and $\mathbf{w}_n \in W$.

4. The Variational Formulation

We are now ready for the variational formulation of the problem. We shall assume that σ is a positive constant, and that the function ζ: $\mathbb{R}^3 \to \mathbb{R}^3$ is uniformly Lipschitz continuous and strongly monotone, that is,

$$\begin{aligned} &|\zeta(u) - \zeta(v)| \leqslant L\,|u - v| \\ &(\zeta(u) - \zeta(v), u - v) \geqslant l\,|u - v|^2 \end{aligned} \qquad l, L > 0;\ \forall u, \forall v \in \mathbb{R}^3.$$

From Eq. (1.9) we deduce that

$$\forall \psi \in \mathscr{U}, \qquad (\nabla \varphi, \nabla \psi) = (\partial A/\partial t, \nabla \psi) \text{ in } \mathbb{L}^2(\Omega) \tag{4.1}$$

which characterizes $\nabla\varphi$ as the projection on K of $\partial A/\partial t$, that is, $\nabla\varphi = M(\partial A/\partial t)$. We shall make a change of variable, considering the unique solution (Duff [4, VI]) β of

$$\left.\begin{aligned} \operatorname{curl}\beta &= 0 \\ \operatorname{div}\beta &= 0 \end{aligned}\right\} \quad \text{in } \Omega;$$

$$\mathbf{n}\cdot\beta = 0 \qquad \text{on } \partial\Omega;$$

$$\int_\Gamma \nu\cdot\beta = \omega,$$

and the unique function $\pi \in H(\operatorname{curl}, \Omega)$ defined by (Duff [3, III]):

$$\operatorname{curl}\pi = \beta,$$
$$\operatorname{div}\pi = 0,$$
$$\mathbf{n}\times\pi = \tilde{\alpha}.$$

We remark that $P = A - \pi \in V_0$; substituting in Eq. (1.10) we obtain (from here on, we shall pose $u' = du/dt$, etc.):

$$\sigma P' + \operatorname{curl}\zeta(\operatorname{curl}P + \beta) = \sigma MP' + \sigma M\pi' - \sigma\pi',$$
$$\sigma NP' + \operatorname{curl}\zeta(\operatorname{curl}P + \beta) = -\sigma N\pi'.$$

Such an equation yields the following variational problem:

PROBLEM I. Find P such that

$$P \in L^2(0, T; V_0), \ P' \in L^2(0, T; H),$$
$$\sigma(NP', \mathbf{v}) + (\zeta(\operatorname{curl}P + \beta), \operatorname{curl}\mathbf{v}) = -\sigma(N\pi', \mathbf{v}), \qquad \forall\mathbf{v}\in V_0, \tag{4.2}$$
$$P(0) = P_0 \ (= A_0 + \pi(0)).$$

We recall now that $N = N^2$, so that, setting $NP = Q \in W$, $N\mathbf{v} = \mathbf{w} \in W$, $-N\pi' = \mathbf{f} \in H_0$, if $\omega(t)$ is suitably regular, and remarking that if $\mathbf{v} \in V_0$, $\operatorname{curl}\mathbf{w} = \operatorname{curl}N\mathbf{v} = \operatorname{curl}\mathbf{v}$, we obtain

PROBLEM II. Find Q such that

$$Q \in L^2(0, T; W); \ Q' \in L^2(0, T; H_0),$$
$$\sigma(Q', \mathbf{w}) + (\zeta(\operatorname{curl}Q + \beta), \operatorname{curl}\mathbf{w}) = \sigma(\mathbf{f}, \mathbf{w}), \qquad \forall\mathbf{w}\in W, \tag{4.3}$$
$$Q(0) = Q_0 \ (= NP_0).$$

Remarking that $W \subseteq H_0 \subseteq W'$ with continuous injections and dense images, existence and uniqueness results for Problem II are assured by

THEOREM I. *Under the hypotheses*: $\mathbf{f}, \mathbf{f}' \in L^2(0, T; H_0)$; $Q_0 \in W$;

$$\operatorname{curl} \zeta(\operatorname{curl} Q_0 + \beta(0)) \in H_0 \tag{4.4}$$

there exists a unique solution Q of Problem II *such that*

$$Q \text{ and } Q' \in L^2(0, T; W) \cap L^\infty(0, T; H_0).$$

Moreover, $\operatorname{curl} \zeta(\operatorname{curl} Q + \beta) \in L^2(0, T; H_0)$.

Remark I. The hypotheses assumed for $\mathbf{f}$ come true if the flux $\omega(t) \in H^2(0, T)$, so that $\partial^k \pi/\partial t^k \in L^2(0, T; V_{\partial^k \alpha/\partial t^k})$, $k = 0, 1, 2$;[2] also, $Q_0 \in W$ and $\operatorname{curl} \zeta(\operatorname{curl} Q_0 + \beta(0)) = j(0)$ imply that $B(0) \in \mathbb{L}^2(\Omega)$.

Remark II. The solution Q assured by Theorem I uniquely determines a function $P = N_0^{-1} Q$, and therefore $A = P + \pi$. Then, the projection MA' uniquely determines a vector $u \in K$ which, by Proposition III, is such that $u = \nabla\varphi$, $\varphi \in \mathscr{V}$. Moreover, $A \in L^2(0, T; V_\alpha)$, $A' \in L^2(0, T; V_{\alpha'})$, $\nabla\varphi \in L^2(0, T; \mathbb{L}^2(\Omega))$.

Proof of Theorem I. Define the operator $\mathscr{N}: W \to W'$ and the functional $\mathbf{y} \in W'$ by

$$\left.\begin{aligned} \langle \mathscr{N}\mathbf{u}, \mathbf{v} \rangle &= (\zeta(\operatorname{curl} \mathbf{u} + \beta) - \zeta(\beta), \operatorname{curl} \mathbf{v}) \\ \langle \mathbf{y}, \mathbf{v} \rangle &= -(\zeta(\beta), \operatorname{curl} \mathbf{v}) \end{aligned}\right. \qquad \forall \mathbf{u}, \forall \mathbf{v} \in W.$$

$\mathscr{N}$ is easily seen to be coercive, strongly monotone and continuous on W, with

$$\begin{aligned} &\langle \mathscr{N}u, u \rangle \geqslant l \, \|u\|_W^2 \\ &\langle \mathscr{N}u - \mathscr{N}v, u - v \rangle \geqslant l \, \|u - v\|_W^2 \qquad \forall u, \forall v \in W; \\ &\|\mathscr{N}u\|_{W'} \leqslant L \, \|u\|_W \end{aligned}$$

we now claim that

PROPOSITION VI. $\mathscr{N}\mathbf{u} \in H_0 \Leftrightarrow \mathbf{h} = \operatorname{curl}(\zeta(\operatorname{curl} \mathbf{u} + \beta) - \zeta(\beta)) \in H_0$.

Proof. $\mathscr{N}\mathbf{u} \in H_0$ means that $\exists g \in H_0$ such that $\forall \mathbf{w} \in W$, $\langle \mathscr{N}\mathbf{u}, \mathbf{w} \rangle = (g, \mathbf{w})$; also, $\forall \mathbf{w} \in W$ there is a uniquely determined $\mathbf{v} \in V_0$ such that $\mathbf{w} = N\mathbf{v}$. Hence: if $\mathbf{h} \in H_0$, since N is the projection of $\mathbb{L}^2(\Omega)$ on H_0, we have

[2] $L^2(0, T; V_{\partial^k \alpha/\partial t^k}) = \{u \in L^2(0, T; H(\operatorname{curl}, \Omega)) \mid \partial^k u/\partial t^k \in V_{\partial^k \alpha/\partial t^k} \text{ a.e.}\}$.

$\langle \mathcal{N}\mathbf{u}, \mathbf{w}\rangle = (\zeta(\operatorname{curl}\mathbf{u}+\beta)-\zeta(\beta),\ \operatorname{curl}\mathbf{w}) = (\zeta(\operatorname{curl}\mathbf{u}+\beta)-\zeta(\beta),\ \operatorname{curl}\mathbf{v}) = (\mathbf{h},\mathbf{v}) = (N\mathbf{h},\mathbf{v}) = (\mathbf{h}, N\mathbf{v}) = (\mathbf{h},\mathbf{w})$, so that taking $g=\mathbf{h}\in H_0$, we may conclude that $\mathcal{N}\mathbf{u}\in H_0$. Conversely, if $\mathcal{N}\mathbf{u}\in H_0$, we remark that $\forall\mathbf{v}\in V_0$, $(\zeta(\operatorname{curl}\mathbf{u}+\beta)-\zeta(\beta), \operatorname{curl}\mathbf{v}) = (\zeta(\operatorname{curl}\mathbf{u}+\beta)-\zeta(\beta), \operatorname{curl} N\mathbf{v}) = \langle \mathcal{N}\mathbf{u}, N\mathbf{v}\rangle = (g, N\mathbf{v}) = (Ng,\mathbf{v}) = (g,\mathbf{v})$. Taking now $\varphi\in(\mathscr{D}(\Omega))^3$, and defining $\mathbf{p}$ by

$$\mathbf{p}\in H_0^1(\Omega),$$
$$-\Delta\mathbf{p} = \operatorname{div}\varphi,$$

so that $\varphi+\nabla\mathbf{p}\in V_0$, we have $\langle\mathbf{h},\varphi\rangle = (\zeta(\operatorname{curl}\mathbf{u}+\beta)-\zeta(\beta), \operatorname{curl}\varphi) = (\zeta(\operatorname{curl}\mathbf{u}+\beta)-\zeta(\beta), \operatorname{curl}(\varphi+\nabla\mathbf{p})) = (g,\varphi+\nabla\mathbf{p})$. But since $g\in H_0$ and $\nabla\mathbf{p}\in K$, $(g,\nabla\mathbf{p})=0$, so that $\langle\mathbf{h},\varphi\rangle=(g,\varphi)$ and $\mathbf{h}=g\in H_0$.

Proposition VI allows us then to restate Eqs. (4.3) and (4.4) by

$$\sigma(Q',\mathbf{w}) + \langle\mathcal{N}Q,\mathbf{w}\rangle = \sigma(\mathbf{f},\mathbf{w}) + \langle\mathbf{y},\mathbf{w}\rangle \qquad \forall\mathbf{w}\in W, \tag{4.5}$$

$$Q(0)=Q_0\in W; \qquad \mathcal{N}Q_0-\mathbf{y}(0)\in H_0. \tag{4.6}$$

Such problem can be handled by a standard Galerkin approximation method, with a total basis in W: we derive here some a priori estimates which are formally written for the whole spaces. We shall note with $\|\cdot\|$ the norm in W and $|\cdot|$ the norm in H_0.

At first, multiplying (4.5) by Q, and referring to standard techniques (Lions [9]) we get

$$\|Q\|_{L^2(0,T;W)} + \|Q\|_{L^\infty(0,T;H_0)} \leqslant \text{const.}$$

Considering then Eq. (4.5) for $t=0$, and multiplying by $Q'(0)$, we get

$$\sigma|Q'(0)|^2 + \langle\mathcal{N}Q_0, Q'(0)\rangle = \sigma(f(0),Q'(0)) + \langle\mathbf{y}(0),Q'(0)\rangle;$$

that is, because of (4.6),

$$\begin{aligned}\sigma|Q'(0)|^2 &= (\sigma\mathbf{f}(0)+\mathbf{y}(0)-\mathcal{N}Q_0, Q'(0))\\ &\leqslant |\sigma\mathbf{f}(0)+\mathbf{y}(0)-\mathcal{N}Q_0|\cdot|Q'(0)|,\end{aligned}$$

whence

$$\sigma|Q'(0)|\leqslant \text{const.} \tag{4.7}$$

Differentiating now Eq. (4.5) and multiplying by Q', we get

$$\sigma(Q'',Q') + \left\langle\frac{d}{dt}\mathcal{N}Q, Q'\right\rangle = \sigma(\mathbf{f}',Q') + \langle\mathbf{y}',Q'\rangle.$$

We recall that ζ, being Lipschitz continuous, has an a.e. defined uniformly bounded derivative ζ', so that we have

$$\begin{aligned}\frac{1}{2}\sigma\frac{d}{dt}|Q'|^2 &+ (\zeta'(\operatorname{curl} Q+\beta)(\operatorname{curl} Q'+\beta'), \operatorname{curl} Q')\\ &- (\zeta'(\beta)\beta', \operatorname{curl} Q')\\ &= \sigma(\mathbf{f}', Q') - (\zeta'(\beta)\beta', \operatorname{curl} Q'),\end{aligned}$$

from which

$$\sigma\frac{d}{dt}|Q'|^2 + c\,\|Q'\|^2 \leqslant 2(\zeta'(\operatorname{curl} Q+\beta)\beta', \operatorname{curl} Q') + 2\sigma(\mathbf{f}', Q');$$

that is, for all $\eta > 0$,

$$\sigma\frac{d}{dt}|Q'|^2 + c\,\|Q'\|^2 \leqslant c_\eta\,|\beta'|^2 + \eta\,\|Q'\|^2 + \sigma\,|\mathbf{f}'|^2 + \sigma\,|Q'|^2.$$

From this we get, as usual, integrating and recalling (4.7),

$$\|Q'\|_{L^2(0,T;W)} + \|Q'\|_{L^\infty(0,T;H_0)} \leqslant \text{const.}$$

The existence of a solution of Problem II can therefore be proved. Moreover, since $Q' \in L^2(0,T;H_0)$, we have that

$$\mathcal{A}Q - \mathbf{y} = \sigma(\mathbf{f} - Q') \in L^2(0,T;H_0).$$

The proof of uniqueness is standard, because of the monotonicity of $\mathcal{A}$.

Remark on periodic solutions. Since the operator $\mathcal{A}$ is monotone and continuous the problem of a periodic solution for Eq. (4.5) can be handled referring to the results of Lions [9, II, 7.4]: we have then the following.

THEOREM II. *Under the hypothesis* $\mathbf{f} \in L^2(0,T;H_0)$, *there exists a unique solution* Q *of the problem*

$$\begin{aligned}\sigma Q' + \mathcal{A}Q &= \sigma\mathbf{f} + \mathbf{y},\\ Q(0) &= Q(T)\end{aligned} \tag{4.8}$$

such that $Q \in L^2(0,T;W)$ *and* $Q' \in L^2(0,T;W')$.

We can obtain greater regularity on Q', making the further assumption that $\mathcal{A} = \partial\varphi$ with φ convex lower semicontinuous function in H_0 (such is the case when $\zeta = \partial F$, with F convex function in $C^{1,1}(\mathbb{R}^3)$; see Negro [10]). From Brézis [1, III, 6, 3.4] we obtain then

THEOREM III. *Under the hypothesis* $\mathbf{f} \in L^2(0, T; H_0)$, *there exists a unique solution* Q *of problem* (4.8) *such that* $Q \in L^2(0, T; W)$ *and* $Q' \in L^2(0, T; H_0)$.

In this case too we obtain $\mathcal{A}Q - \mathbf{y} \in L^2(0, T; H_0)$.

5. A DIRECT APPROACH TO THE PROBLEM

We remark that a direct formulation for Eqs. (1.10) and (4.1) can be considered, which may be more useful for the numerical treatment of the problem. Equations (1.10) and (4.1) yield the following.

PROBLEM III. Find A, φ such that

$$A \in L^2(0, T; V_\alpha); \qquad A' \in L^2(0, T; H); \qquad \varphi \in L^2(0, T; \mathscr{U}); \qquad A(0) = A_0;$$

$$\sigma(A' - \nabla\varphi, \mathbf{v}) + (\zeta(\operatorname{curl} A), \operatorname{curl} \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in V_0, \tag{5.1}$$

$$(\nabla\varphi, \nabla\psi) = (A', \nabla\psi) \qquad \forall \psi \in \mathscr{U}. \tag{5.2}$$

The existence and uniqueness Theorem for Problem III is stated as

THEOREM Ibis. *Under the hypotheses*

$$\alpha \in H^2(0, T), \qquad A(0) = A_0 \in V_{\alpha(0)}, \operatorname{curl} \zeta(\operatorname{curl} A_0) \in H_0,$$

there exists a unique solution A, φ *of Eqs.* (5.1) *and* (5.2) *such that*

$$A \in L^2(0, T; V_\alpha) \cap L^\infty(0, T; H); \qquad A' \in L^2(0, T; V_{\alpha'}); \qquad \varphi \in L^2(0, T; \mathscr{U}).$$

A direct proof of this theorem can be given, by a Galerkin approximation method, using the same functions β and π defined in Section 4, and obtaining analogous a priori estimates. In fact, differentiating Eqs. (5.1) and (5.2), and letting $\mathbf{v} = A' - \pi'$ and $\psi = \varphi'$, it is possible to obtain, because of the assumptions on A_0, the estimates

$$\|A' - \nabla\varphi\|_{L^\infty(0, T; \mathbb{L}^2(\Omega))} + \|A'\|_{L^2(0, T, H(\operatorname{curl}, \Omega))} + \|\varphi\|_{L^2(0, T; \mathscr{U})} \leqslant \text{const.}$$

Letting then $v = A - \pi$, we also get

$$\|A\|_{L^2(0, T; H(\operatorname{curl}, \Omega))} + \|A\|_{L^\infty(0, T; H)} \leqslant \text{const.}$$

As for uniqueness, if $A^* = A_1 - A_2 \in V_0$ and $\varphi^* = \varphi_1 - \varphi_2$, we get, by subtraction,

$$\frac{d}{dt}|A^*|^2 + 2(\zeta(\operatorname{curl} A_1) - \zeta(\operatorname{curl} A_2), \operatorname{curl} A^*) = 2(\nabla\varphi^*, A^*). \tag{5.3}$$

Choosing

$$\psi(t) = \int_0^t \varphi^*$$

so that $\psi(0) = 0$ and $(\nabla\psi', \nabla\psi) = (A^{*\prime}, \nabla\psi)$, we get

$$2\int_0^T (A^{*\prime}, \nabla\psi) = |\nabla\psi(T)|^2,$$

and integrating by parts

$$2\int_0^T (A^*, \nabla\psi') + |\nabla\psi(T)|^2 = 2(A^*(T), \nabla\psi(T)) - 2(A_0^*, \nabla\psi(0))$$

from which

$$2\int_0^T (A^*, \nabla\varphi) \leqslant |A^*(T)|^2.$$

Equation (5.3) yields then

$$|A^*(T)|^2 + c\int_0^T |\operatorname{curl} A^*|^2 \leqslant |A^*(T)|^2,$$

that is to say,

$$|\operatorname{curl} A^*| = 0.$$

A^* is therefore a solution of

$$\operatorname{curl} A^* = 0,$$

$$\operatorname{div} A^* = 0,$$

$$n \times A^* = 0$$

and so (Duff [4, VI]) $A^* = 0$ a.e.; moreover, since

$$|\nabla\varphi^*| \leqslant |A^{*\prime}|,$$

we also have $|\nabla\varphi^*| = 0$, that is, $\varphi^* = 0$ a.e.

Remark. In the case when $\sigma = \sigma(x)$ is a positive bounded measurable function in Ω, that is, when

$$0 < s \leqslant \sigma(x) \leqslant S < +\infty \text{ a.e.,}$$

a priori estimates analogous to those obtained in this section can be established, permitting one to handle the problem. A transformation into a unique parabolic equation, such as carried out in Section 4, is slightly more complicated, and would require the use of analogous Sobolev spaces with weights.

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